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COMMENT

The mKdV with one-half degree of nonlinearity is *not* integrable

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Abstract. We show that the 'mKdV with one-half degree of nonlinearity', proposed by Xiao, does not pass the Painlevé test, contrary to the author's claim.

In a recent letter, Yi Xiao [1] proposed a new modified-KdV equation of the form:

$$16u_t + u_{xxx} + 30u^{1/2}u_x = 0 \tag{1}$$

which according to his calculations should be integrable. His conclusion was based on a Painlevé analysis of the equation:

$$n^2u_t + u_{xxx} + (n+1)(n+2)u^{2/n}u_x = 0 \tag{2}$$

which he transformed, through $u = q^n$, into:

$$n^2q^2q_t + (n-1)(n-2)q_x^3 + 3(n-1)qq_xq_{xx} + q^2q_{xxx} + (n+1)(n+2)q^4q_x = 0. \tag{3}$$

The singular behaviour $q \sim \phi^{-1}$ (where $\phi(x, t) = 0$ is the singularity manifold) was considered, and the author of [1] has shown that, for $n = 4$ (in which case (2) reduces to (1)) this leading behaviour possessed integer resonances and a pure Laurent expansion. He thus concluded that (1) had the Painlevé property. On the basis of the well known conjecture [2-4], he deduced the integrability of this equation.

Unfortunately this conclusion is wrong: this equation does *not* possess the Painlevé property. As is often the case with Painlevé analyses leading to unexpected results, the author just missed one possible leading behaviour, which does not pass the Painlevé test. Let us illustrate this using (2) for general n. One possible leading behaviour is the following: $q \sim \phi^1$. In this case u_t and u_{xxx} are generically finite. The resonances are -1, 0 and 1. The rightmost term in (2) is non-dominant. For $n = 1$ or 2 this term is regular, and the whole expansion is just around a perfectly regular zero, which is why it is generally not even considered as a 'potentially singular' behaviour for KdV and mKdV. However, when $n > 2$, this term, though subdominant, is singular. It generates a term with a fractional power $\phi^{3+2/n}$ in the expansion, making it non-Painlevé. For instance for $n = 4$, a singular behaviour would begin as (taking the Kruskal ansatz [5], $\phi = x + \psi(t)$, $a_i = a_i(t)$):

$$u = \phi(a_0 + a_1\phi - 8a_0\psi_t\phi^2/3 - 16a_0^{3/2}\phi^{5/2}/7 + \ldots)$$

with a_0, a_1 free functions of t. The term $\phi^{7/2}$ is the one destroying the Painlevé property and thus spoils the hopes for the integrability of (1).

We would have reached the same conclusion by analysing (3) directly. There, a behaviour $q \sim \phi^{1/n}$ does exist. The only dominant terms for this behaviour are the three arising from u_{xxx}. The expansion is not of the form $\phi^{1/n}$ times a Taylor series in ϕ. Indeed, the rightmost term $q^4 q_x$ is smaller than the dominant ones by a factor $\phi^{2+2/n}$. Unless $n = 1$ or 2 this is not an integer power of ϕ and therefore (3) does not pass the Painlevé test either, with this singular leading behaviour.

References

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